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Zero Bounds for a Certain Class of Polynomials

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Abstracts

In this paper we give a bound for the zeros of a polynomial with complex coefficients. Our results generalize some known results in addition to giving a way for some new results.

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Keywords: Enestrom – Kakeya Theorem, Zeros, Bounds.

Introduction

An elegant result in the theory of distribution of zeros of polynomials is the following theorem due to Enestrom and Kakeya [10]:

Theorem A: If the coefficients of the polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfy $0 \le a_0 \le a_1 \le ...... \le a_{n-1} \le a_n$, then all the zeros of P(z) lie in the closed disk $|z| \le 1$.

In the literature ([2], [4]-[6], [8]-[12]) there exist several generalizations of this result. Aziz and Mohammad [1] proved the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 \ge t_2 \ge 0$ can found such that

$$a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \ge 0,\ j = 1,2,......, n+1 (a_{-1} = a_{n+1} = 0),$$

then all the zeros of P(z) lie in $|z| \le t_1$.

For $t_1 = 1, t_2 = 0$, it reduces to Theorem A.

Aziz and Shah [3] proved the following more general result which includes Theorem A as a special case:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some t>o,

$$\max_{|z|=R} \left| ta_0 z^n + (ta_1 - a_0) z^{n-1} + ... + (ta_n - a_{n-1}) \right| \le M$$

where R is any positive real number, then all the zeros of P(z) lie in

$$|z| \le \max\left(\frac{M}{|a_n|}, \frac{1}{R}\right).$$

Recently B. A. Zargar [13] proved a more general result which includes Theorem A as a special case. He proved

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some real numbers $t_1, t_2; t_1 \ne 0, t_1 > t_2 \ge 0$,

$$\max_{|z|=R} \left| \sum_{j=0}^{n} \{a_j t_1 t_2 + a_{j-1}(t_1 - t) - a_{j-2}\} z^{n-j} \right| \le M,$$

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where R is a positive real number. Then, all the zeros of P(z) lie in

$$|z| \le \max(r, \frac{1}{R}),$$

where

$$r = \frac{2M}{\{|a_n(t_1 - t_2) - a_{n-1}|^2 + 4|a_n|M\}^{\frac{1}{2}} - |a_n(t_1 - t_2) - a_{n-1}|}.$$

Main results

In this paper we prove a generalization of Theorem D as follows:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j = \alpha_j + i\beta_j, \quad j = 0,1,2,......,n$ such that for some real numbers $t_1, t_2; t_1 \ne 0, t_1 > t_2 \ge 0$,

$$\max_{|z|=R} \left| \sum_{j=0}^{n} \{\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t) - \alpha_{j-2}\} z^{n-j} \right| \le M_1,$$

$$\max_{|z|=R} \left| \sum_{j=0}^{n} \{\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t) - \beta_{j-2}\} z^{n-j} \right| \le M_2,$$

where R is a positive real number. Then, all the zeros of P(z) lie in

$$|z| \le \max(r_1, \frac{1}{R}),$$

where

$$r_1 = \frac{2(M_1 + M_2)}{\{|a_n(t_1 - t_2) - a_{n-1}|^2 + 4|a_n|(M_1 + M_2)\}^{\frac{1}{2}} - |a_n(t_1 - t_2) - a_{n-1}|}.$$

Remark 1: Taking $\beta_j = 0, \forall j = 0,1,2,......,n,$, we get Theorem D.

Taking $t_2 = 0$, we get the following generalization of a result of Zargar [13, Cor.4] which includes Theorem A as a special case.

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $a_j = \alpha_j + i\beta_j, \quad j = 0,1,2,......,n$ such that for some real number $t > 0$,

$$\max_{|z|=R} \left| \sum_{j=0}^{n} (\alpha_{j-1} t - \alpha_{j-2}) z^{n-j} \right| \le M_1,$$

$$\max_{|z|=R} \left| \sum_{j=0}^{n} (\beta_{j-1} t - \beta_{j-2}\} z^{n-j} \right| \le M_2,$$

where R is a positive real number. Then, all the zeros of P(z) lie in

$$|z| \le \max(r_1, \frac{1}{R}),$$

where

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$$r_1 = \frac{2(M_1+M_2)}{\{|a_n t - a_{n-1}|^2 + 4|a_n|(M_1+M_2)\}^{\frac{1}{2}} - |a_n t - a_{n-1}|}.$$

Lemmas

For the proofs of the above results , we need the following results:

Lemma 1: If f(z) is analytic for $|z| \le 1$, f(0)=a, where $|a| < 1$, $f'(0) = b$, $|f(z)| \le 1$ for $|z| = 1$, then

$$|f(z)| \le \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

The inequality is sharp with equality for the function

$$f(z) = \frac{a + \frac{b}{1+a} z - z^2}{1 - \frac{b}{1+a} z - az^2}.$$

The above lemma is due to Govil, Rahman and Schmeisser [7].

Lemma 2: If f(z) is analytic for $|z| \le R$, f(0)=0, , $f'(0) = b$, $|f(z)| \le M$ for $|z| = R$, then

$$|f(z)| \le \frac{M|z|}{R^2} . \frac{M|z| + R^2|b|}{M + |b||z|} \text{ for } |z| \le R .$$

Lemma 2 is a simple deduction from Lemma 1.

Proofs of theorems

Proof of Theorem 1: Consider the polynomial

$$F(z) = (t_2 + z)(t_1 - z)P(z) = (t_2 + z)(t_1 - z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+2} + \{a_n(t_1 - t_2) - a_{n-1}\} z^{n+1} + \sum_{j=0}^{n} \{a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2}\} z^j$$

$$= -a_n z^{n+2} + \{\alpha_n(t_1 - t_2) - \alpha_{n-1}\} z^{n+1} + \sum_{j=0}^{n} \{\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2}\} z^j$$

$$+ i[\{\beta_n(t_1 - t_2) - \beta_{n-1}\} z^{n+1} + \sum_{j=0}^{n} \{\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2}\} z^j].$$

Let $G(z) = z^{n+2} F(\frac{1}{z})$

$$= -a_n + \{a_n(t_1 - t_2) - a_{n-1}\} z + \sum_{j=0}^{n} \{\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2}\} z^{n-j+2}$$

$$+ i \sum_{j=0}^{n} \{\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2}\} z^{n-j+2}]$$

$$= -a_n + \{a_n(t_1 - t_2) - a_{n-1}\} z + H(z),$$

where

$$H(z) = z^2 [\sum_{j=0}^{n} \{\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2}\} z^{n-j} + i \sum_{j=0}^{n} \{\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2}\} z^{n-j}].$$

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Then H(0)=0= $H'(0)$ and by using the hypothesis $|H(z)| \le R^2(M_1 + M_2)$ for $|z| = R$.

We first suppose that

$$|a_n| \le R^2(M_1 + M_2) + R|a_n(t_1 - t_2) - a_{n-1}|.$$

Then by applying Lemma 2 to H(z) we get

$$|H(z)| \le \frac{R^2(M_1 + M_2)|z|}{R^2} \cdot \frac{R^2(M_1 + M_2)|z|}{R^2(M_1 + M_2)} = (M_1 + M_2)|z|^2 .$$

Hence

$$\begin{aligned} |G(z)| &= |-a_n + \{a_n(t_1 - t_2) - a_{n-1}\}z + H(z)| \\ &\ge |a_n| - |a_n(t_1 - t_2) - a_{n-1}||z| - |H(z)| \\ &\ge |a_n| - |a_n(t_1 - t_2) - a_{n-1}||z| - (M_1 + M_2)|z|^2 \\ &> 0 \end{aligned}$$

if

$$(M_1 + M_2)|z|^2 + |a_n(t_1 - t_2) - a_{n-1}||z| - |a_n| < 0$$

i.e. if

$$\begin{aligned} |z| &< \frac{\{|a_n(t_1 - t_2) - a_{n-1}|^2 + 4|a_n|(M_1 + M_2)\}^{\frac{1}{2}} - |a_n(t_1 - t_2) - a_{n-1}|}{2(M_1 + M_2)} \\ &= \frac{1}{r} \le R \end{aligned}$$

if

$$|a_n(t_1 - t_2) - a_{n-1}|^2 + 4|a_n|(M_1 + M_2) \le \{2(M_1 + M_2)R + |a_n(t_1 - t_2) - a_{n-1}|\}^2$$

or

$$|a_n| \le R^2(M_1 + M_2) + R|a_n(t_1 - t_2) - a_{n-1}|$$

which is true by our supposition.

Thus, it follows that all the zeros of G(z) lie in $|z| \ge \frac{1}{r}$.

Since $F(z) = z^{n+2}G(\frac{1}{z})$, it follows that all the zeros of F(z) lie in $|z| \le r$.

We now suppose that $|a_n| > R^2(M_1 + M_2) + R|a_n(t_1 - t_2) - a_{n-1}|$.

Then , for $|z| \le R$, we have

$$\begin{aligned} |G(z)| &\ge |a_n| - |a_n(t_1 - t_2) - a_{n-1}||z| - |H(z)| \\ &\ge |a_n| - |a_n(t_1 - t_2) - a_{n-1}|R - R^2(M_1 + M_2) \\ &> 0 \end{aligned}$$

This shows that G(z) has all its zeros in $|z| > R$.

Since $F(z) = z^{n+2}G(\frac{1}{z})$, it follows that all the zeros of F(z) lie in $|z| < \frac{1}{R}$.

But the zeros of P(z) are also the zeros of F(z). Hence, it follows that all the zeros of P(z) lie in $|z| < \frac{1}{R}$.

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Combining the above two arguments, it follows that all the zeros of P(z) lie in $|z| \leq \max(r, \frac{1}{R})$.

That completes the proof of Theorem 1.

References

[1] A. Aziz and Q.G. Mohammad, On the Zeros of Certain Class of Polynomials and related Analytic Functions, J. Math. Anal. Appl. 75 (1980), 495-502.
[2] A. Aziz and W. M. Shah, On the Location of Zeros of Polynomials and related Analytic Functions, Non-linear Studies , 6(1999), 97-104.
[3] A. Aziz and W. M. Shah, On the Zeros of Polynomials and related Analytic Functions, Glasnike Mathematicke, 33(1998), 173-184.
[4] A. Aziz and B. A. Zargar, Some Extensions of Enestrom-Kakeya Theorem, Glasnike Mathematicke, 31(1996), 239-244.
[5] K. K. Dewan and M. Bidkham, On the Enestrom-Kakeya Theorem, J. Math. Anal. Appl. 180(1993), 29-36.
[6] N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math. J. 20(1968), 126-136.
[7] N. K. Govil, Q. I. Rahman and G. Schmessier, On the Derivatives of Polynomials, Illinois Math. Journal 23 (19790, 319-329.
[8] M. H. Gulzar, Some Refinements of Enestrom-Kakeya Theorem, International Journal of Mathematical Archive, Vol.2(9), 2011, 1512-1519.
[9] P. V. Krishnaliah, On Kakeya Theorem, J. London Math. Soc. 20(1955), 314-319.
[10] M. Marden, Geometry of Polynomials, Mathematical Surveys No.3 Providence R.I., 1966.
[11] G. V. Milovanovic, D. S. Mitrinovic and T. M. Rassias, Topics in Polynomials , Extremal Problems, Inequalities, Zeros, World Scientific, Singapore, 1994.
[12] Q. I. Rahman and G. Schmessier, Analytic Theory of Polynomials, Clarantone Press, Oxford, 2002.
[13] B. A. Zargar, On the Zeros of Certain Class of Polynomials, International Journal of Modern Engineering Research, Vol.2, Issue 6, Nov.-Dec.20124363-4372.